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ON COMPARABILITY OF IDEALS OF COMMUTATIVE RINGS

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INTRODUCTION

Through out this paper R denotes a commutative ring with identity, Z(R) denotes the set of zerodivisors of R, and N denotes the set of nonunit elements of R. Let A be a quasi-local domain with maximal ideal M and quotient field K. David Anderson [1] studied several comparability conditions between M and certain fractional ideals of A. Our main purpose is to generalize the study of comparability of ideals to the context of arbitrary rings with Z(R) possibly nonzero.

It is fair to state that the whole work in this paper is motivated by [1] and [2].

SECTION 1

In this section we consider the following three comparability conditions that are analogs of those from [1, P. 453] :

(I) For every $a,b \in R$, $aR \subset bR$ or $bR \subset aR$.

(II) For every $a,b \in R$, $aR \subset bN$ or $bN \subset aR$.

(III) For every $a,b \in R$, $aN \subset bN$ or $bN \subset aN$.

Clearly (I) $\Longrightarrow$ (II) $\Longrightarrow$ (III). Note that even for integral domains none of the implications is reversible, for examples see [1]. It is well-known that (I) is an equivalent condition for R to be a chained ring (a valuation ring).

The following theorem is an important tool in our work. The first part is taken from [2, Theorem 1] and the second part is [3, Theorem 2]. Recall from [3], a ring R is called a pseudo-valuation ring (PVR) in case each prime ideal P of R is strongly prime, in the sense that aP and bR are comparable for each $a,b \in R$.

Theorem 0. (1) If for each $a,b \in R$ either $b|a^2$ or $a|b^2$, then the prime ideals of R are linearly ordered and therefore R is quasilocal.

(2) R is a PVR if and only if it is quasi-local with maximal ideal M strongly prime.

Proof. We just provide the proof of (1) since it is so short. Suppose that there are two prime ideals P,Q of R that are not comparable. Let $b \in P\backslash Q$ and $a \in Q\backslash P$. Then neither $b \nmid a^2$ nor $a \nmid b^2$, a contradiction. ■

In the following theorem we show that (II) is an equivalent condition for R to be a pseudo valuation ring (PVR).

Theorem 1. A ring R satisfies (II) if and only if R is a PVR.

Proof. Suppose that R satisfies (II). Let $a,b \in R$. Then $b|a^2$ or $a|b^2$. Hence, the prime ideals of R are linearly ordered by Theorem 0 (1). In particular, R is quasi-local with the maximal ideal N. Thus, R is a PVR by Theorem 0 (2).

For the converse, suppose that R ia a PVR. By [3, Lemma 1] R is quasi-local with the maximal ideal N. Hence, aN and bR are comparable for every $a,b \in R$. ■

In the next theorem, we show that if R satisfies (III) and N contains a non-zerodivisor, then R is quasi-local with the maximal ideal N and $N:N = \{ x \in T : xN \subset N \}$ is a chained ring (valuation ring), where $T = R_S$ is the total quotient ring of R and S is the set of non-zerodivisors of R. Observe that if R satisfies (II) and N contains a non-zerodivisor of R, then N:N is a chained ring with maximal ideal N by [3, Theorem 8]. This shows that the distinction between (II) and (III) is whether or not N is a maximal ideal of N:N (see [1, Example 3.2]).

Theorem 2. Suppose that R satisfies (III). Then

(1) R is quasi-local with maximal ideal N.

(2) If N contains a non-zerodivisor element, then R is quasi-local with maximal ideal N and N:N is a chained ring.

Proof. (1). Let $a,b \in R$. Then $b|a^2$ or $a|b^2$ Once again, by Theorem 0 (1) R is quasi-local with maximal ideal N. **(2).** Suppose that N contains a non-zerodivisor element. Now, let s be a non zerodivisor element in N and $x,y \in N:N$. Then $x = a/d$ and $y = b/d$ for some $a,b \in R$ and a non-zerodivisor d of R. Since aN and bN are comparable, we may assume that $aN \subset bN$. Thus, $as = bk$ for some $k \in N$, and therefore in N:N we have $(a/d)s = (b/d)k$. We consider two cases : *case 1.* Suppose that $k \in Z(R)$. Then $kN \subset sN$, for otherwise $k|s^2$ which is impossible since s is non-zerodivisor. Thus $k/s \in N:N$ and $y|x$ in N:N. *Case 2.* Suppose that $k \notin Z(R)$. Then $k/s \in N:N$ or $s/k \in N:N$ and hence $y|x$ in N:N or $x|y$ in N:N. Thus, N:N is a chained ring. ■

Example 10 (a) in [3] shows that the non-zerodivisor hypothesis is needed in Theorem 2 (2).

Our next result is motivated by [1, Proposition 3.3].

Theorem 3. Assume that for each $a,b \in R$, there is a maximal ideal M of R containing $Z(R)$ so that aM and bM are comparable. Then the prime ideals of R are linearly ordered. In particular, R is quasi-local.

Proof. Assume that R has two distinct maximal ideals M and L. Choose $a \in M\backslash L$ and $b \in L\backslash M$. By hypothesis there is a maximal ideal P of R containing $Z(R)$ so that $aP \subset bP$ or $bP \subset aP$. If $aP \subset bP$, then $aP \subset bP \subset L$. Thus, $P \subset L$ and hence $L = P$ since P is maximal. Hence, $ab = bk$ for some $k \in L$ since $aP \subset bP$ and $P = L$ and $b \in L$. Hence, $b(a-k) = 0$ and therefore $a-k \in Z(R) \subset L$. Thus, $a \in L$, a contradiction. If $bP \subset aP$, then we leave this case for the reader to find a contradiction. Thus, R is quasi-local with the maximal ideal N. Now, since for each $a,b \in R$ aN and bN are comparable, either $b|a^2$ or $a|b^2$. Thus, the prime ideals of R are linearly ordered by Theorem 0 (1). ∎

The ring $R = Z/2Z \times Z/2Z$ shows that the hypothesis $Z(R) \subset M$ is needed in Theorem 3.

In light of the above results, we state the following corollary (see [1, Corollary 3.4]):

Corollary 4. The following statements are equivalent :

(1) R satisfies (III).

(2) For some maximal ideal M of R containing $Z(R)$, aM and bM are comparable for each $a,b \in R$.

(3) For each $a,b \in R$, there is a maximal ideal M of R containing $Z(R)$ so that aM and bM are comparable.

We ask the reader to compare our next result with [1, Proposition 3.5] :

Theorem 5. Assume that for each $a,b \in R$ there is a maximal ideal M of R containing $Z(R)$ so that aR and bM are comparable. Then R is a PVR.

Proof. Assume that R has two distinct maximal ideals M and L. Choose $a \in M\setminus L$ and $b \in L\setminus M$. By hypothesis there is a maximal ideal P of R containing $Z(R)$ so that $a^2P \subset b^2R$ or $b^2R \subset a^2P$. If $a^2P \subset b^2R$, then $a^2P \subset b^2R \subset L$. Thus, $P \subset L$ since $a^2 \notin L$. Hence, $P = L$ since P is maximal. Since $P = L$ and $b \in L$ and $a^2b = b^2k$ for some $k \in R$, $b(a^2 - bk) = 0$. Thus, $a^2 - bk \in Z(R) \subset L$. Since $bk \in L$, $a \in L$, a contradiction. If $b^2R \subset a^2P$, then $b^2R \subset a^2P \subset M$.

Thus, $b \in M$, a contradiction. Hence, R is quasi-local. By Theorem 0 (2) R is a PVR. ■

Example 6. Let F and K be any fields. The ring $R = F \times K$ shows that the hypothesis $Z(R) \subset M$ is needed in Theorem 5.

Now we state the following corollary :

Corollary 7. The following statements are equivalent :

(1) R is a PVR (and thus quasi-local).

(2) For each $a, b \in R$ and maximal ideal M of R, aM and bR are comparable.

(3) For some maximal ideal M of R containing $Z(R)$, aM and bR are comparable for each $a, b \in R$.

(4) For each $a, b \in R$, there is a maximal ideal M of R containing $Z(R)$ so that aM and bR are comparable.

SECTION 2

In this section we consider the following comparability condition :

(i) For each $a, b \in R$, $aN \subset bR$ or $bN \subset aR$.

Observe that (i) is the analog of (ii from [1, P.454]. Also, observe that if (i) holds, then either $b|a^2$ or $a|b^2$, so the prime ideals of R are linearly ordered.

We have the following (see [1, Proposition 3.7]):

Theorem 8. Assume that for each $a,b \in R$, there is a maximal ideal M of R containing Z(R) so that $aM \subset bR$ or $bM \subset aR$. Then the prime ideals of R are linearly ordered. In particular, R is quasi-local.

Proof. The proof is essentially the same as in Theorem 5. So we leave the proof to the reader. Hence R is quasi-local with the maximal ideal N. Since for each $a,b \in R$ either $a|b^2$ or $b|a^2$, by Theorem 0 (1) the prime ideals of R are linearly ordered. ■

Again, example 6 above shows that the $Z(R) \subset M$ hypothesis is needed in Theorem 8.

In view of Theorem 8, we have the following :

Corollary 9. The following statements are equivalent :

(1) R satisfies **(i)**.

(2) The prime ideals of R are linearly ordered and satisfies **(i)**.

(3) R is quasi-local and satisfies **(i)**.

(4) For some maximal ideal M of R containing $Z(R)$, $aM \subset bR$ or $bM \subset aR$ for every $a, b \in R$.

(5) For each $a, b \in R$, there is a maximal ideal M of R containing $Z(R)$ so that $aM \subset bR$ or $bM \subset aR$.

Our last result is a generalization of [1, Proposition 3.10].

Theorem 10. For a ring R, conditions (III) and (i) are equivalent.

Proof. We need only show (i) $\Longrightarrow$ (III). Let $a, b \in R$ so that $aN \subset bR$ and $aN \not\subset bN$. Then for some $s \in N$, $as = b$. Hence, $bN \subset aN$. Similarly, if $bN \subset aR$ and $bN \not\subset aN$, then $aN \subset bN$. ∎

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